Name of College – S.S. College J-Bad

Subject – Mathematics

Topic – Partial Differentiation (Euler's theorem)

Class – B.Sc I (Hons)

Time – 11.40 A.M to 1.20 P.M

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Homogenous Functions

If $u = f(x,y)$ be any Function of x and y. If the sum of powers of x and y in each Term of $f(x,y)$ be equal, then $u = f(x,y)$ is called Homogeneous function.

Example

$$f(x,y) = x^2 + xy + y^2$$
$$f(x,y) = x^2 + y^2$$
$$f(x,y) = x^3 + y^3 + 3x^2y$$
$$f(x,y) = x^4 + 4x^2y^2 + y^4$$

Order of Homogeneous ~~Equation~~ function :→

If the sum of x and y of $f(x,y)$ in each Term is n then $f(x,y)$ is homogenous ~~Eq~~ Function of order n.

$$f(x,y) = x^2 + xy + y^2$$

Here sum of powers of x and y in each Term is 2. Therefore $f(x,y)$ is homogeneous function of x and y of order 2.

it can also be written as.

$$f(x,y) = x^2\left[1 + y/x + \frac{y^2}{x^2}\right]$$
$$= x^2 g(y/x)$$ where g is homogenous function of y/x of order 2.

Euler's theorem on Homogeneous Function of two variables :→

Statement :→ If $u = f(x, y)$ be a homogenous function in x and y of degree n then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Proof :→ Let $u = f(x, y)$

$$= A_1 x^{\alpha_1} y^{\beta_1} + A_2 x^{\alpha_2} y^{\beta_2} + A_3 x^{\alpha_3} y^{\beta_3} + \cdots + A_n x^{\alpha_n} y^{\beta_n} \quad -(1)$$

where $\alpha_1 + \beta_1 = \alpha_2 \beta_2 = \alpha_3 + \beta_3 = \cdots = \alpha_n + \beta_n = n$

Now Diff. (1) partially with respect to x

$$\frac{\partial u}{\partial x} = A_1 \alpha_1 x^{\alpha_1 - 1} y^{\beta_1} + A_2 \alpha_2 x^{\alpha_2 - 1} y^{\beta_2} + \cdots + A_n \alpha_n x^{\alpha_n - 1} y^{\beta_n}$$

$$x\frac{\partial u}{\partial x} = A_1 \alpha_1 x^{\alpha_1} y^{\beta_1} + A_2 \alpha_2 x^{\alpha_2} y^{\beta_2} + \cdots + A_n \alpha_n x^{\alpha_n} y^{\beta_n} \quad -(11)$$

Again Diff. (1) partially with respect to y

$$\frac{\partial u}{\partial y} = A_1 \beta_1 x^{\alpha_1} y^{\beta_1 - 1} + A_2 \beta_2 x^{\alpha_2} y^{\beta_2 - 1} + \cdots + A_n \beta_n x^{\alpha_n} y^{\beta_n - 1}$$

$$y\frac{\partial u}{\partial y} = A_1 \beta_1 x^{\alpha_1} y^{\beta_1} + A_2 \beta_2 x^{\alpha_2} y^{\beta_2} + \cdots + A_n \beta_n x^{\alpha_n} y^{\beta_n} \quad -(111)$$

Adding (11) and (111)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = A_1 x^{\alpha_1} y^{\alpha_1} (\alpha_1 + \beta_1) + A_2 x^{\alpha_2} y^{\beta_2} (\alpha_2 + \beta_2) + \cdots + A_n x^{\alpha_n} y^{\beta_n} (\alpha_n + \beta_n)$$

Since $\alpha_1 + \beta_1 = \alpha_2 + \beta_2 = \cdots = \alpha_n + \beta_n = n$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Generalisation:→

If $u = f(x, y, z)$ be a homogenous function of order n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

Ex:→ $u = x^2 + xy + y^2$

then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$

Here $u = x^2 + xy + y^2$

u is the function of x and y of order 2.

$\frac{\partial u}{\partial x} = 2x + y$

$x\frac{\partial u}{\partial x} = 2x^2 + xy$

$\frac{\partial u}{\partial y} = x + 2y$

$y\frac{\partial u}{\partial y} = xy + 2y^2$

therefore

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2x^2 + xy + xy + 2y^2$$
$$= 2x^2 + 2xy + 2y^2$$
$$= 2(x^2 + xy + y^2)$$
$$= 2u$$

thus

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$$